

Wick ordering for coherent state quantization in $1+1$ de Sitter space

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Abstract

We show that the coherent state quantization of massive particles in $1+1$ de Sitter space exhibits an ordering property: There exist some classical observables A and A^* such that $O_{A^*p}O_{Aq} = O_{A^*pAq}$ $p, q \in \mathbb{Z}$, where O_A is the quantum observable corresponding to the classical observable A .

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1 Introduction

A quantization procedure [1] consists in a map $A \rightarrow O_A$ which associates an operator (quantum observable) O_A to any function A on phase space (classical observable). Now, when two classical observables are quantized an additional ordering rule is needed in order to quantize their product. As a matter of fact the usual methods of quantization, including the most advanced ones like, for instance, geometric quantization[2], allow to quantize only a restricted set of classical observables and do not provide any ordering rule. For instance, the generators of some Lie algebra may be quantized, but even the simplest functions of them like polynomials must be considered separately.

On the other hand, coherent states and their generalizations [3] allow to quantize any classical observable. In addition, in the Bargman representation, the coherent state (or anti-Wick) quantization corresponds to the following ordering : one puts the \bar{z} terms on the left and the z terms on the right and then quantizes. At the quantum level this corresponds to the

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anti-normal ordering in which the derivation terms are on the left. As far as we know, there is no prescription for ordering operators in the general coherent states context, and it could be interesting to exhibit such a rule.

As a matter of fact, and this is the object of the present letter, the coherent states builded on the 1+1 de Sitter phase space for massive particles [4] present such an ordering property: The classical observables which can be expanded as a power series of the two functions $A(\beta, J) = e^{\varepsilon J + i\beta}$ and $A^*(\beta, J)$ are related to the operators O_A and O_{A^*} which verify $O_{A^*p} O_{Aq} = O_{A^*p Aq}$ $p, q \in \mathbb{Z}$. It is important to note that these coherent states are closely related to the coherent states for the motion of massive particle on the circle [5, 6, 7, 8, 9].

In sect. 2 we briefly recall the quantization procedure in the framework of generalized coherent states. The application to the massive free particle in 1 + 1 de Sitter space is summarized in sect. 3. The ordering property is proved in sect. 4.

2 General coherent states

Let X be a set equipped with some measure μ and \mathcal{H} be a separable sub-Hilbert space of $L^2(X, \mu)$. A set of coherent states, and the associated quantization, can be defined if there exists a continuous mapping

$$X \ni x \longrightarrow |x\rangle \in \mathcal{H}, \quad (1)$$

where the family of states $\{|x\rangle\}_{x \in X}$ obeys the following two conditions:

1. *Normalization* : $\langle x | x \rangle = 1$,
2. *Resolution of the identity in \mathcal{H}* : $\int_X |x\rangle \langle x| \nu(dx) = \text{Id}_{\mathcal{H}}$,

where $\nu(dx)$ is another measure on X , absolutely continuous with respect to $\mu(dx)$: there exists a positive measurable function $h(x)$ such that $\nu(dx) = h(x)\mu(dx)$.

In this framework, the coherent states quantization of a *classical* observable, that is to say of a function f on X , consists in associating to f the operator

$$O_f \equiv \int_X f(x) |x\rangle \langle x| \nu(dx). \quad (2)$$

In this context, f is said to be the upper (or contravariant) symbol of the operator O_f , whereas the mean value $\langle x | O_f | x \rangle$ is said to be the lower (or covariant) symbol of O_f [10]. Of course, such a particular quantization

scheme is intrinsically limited to all those classical observables for which the expansion (2) is mathematically justified within the theory of operators in Hilbert spaces (*e.g.* weak convergence).

In practice, the states $|x\rangle$ can be obtained [4] from some superposition of elements of an orthonormal basis $\{|\phi_n\rangle\}_{n\in\mathbb{N}}$ of \mathcal{H} if we assume in addition that

$$\mathcal{N}(x) \equiv \sum_n |\phi_n(x)|^2 < \infty \text{ almost everywhere.} \quad (3)$$

Then, the states

$$|x\rangle \equiv \frac{1}{\sqrt{\mathcal{N}(x)}} \sum_n \phi_n^*(x) |\phi_n\rangle, \quad (4)$$

are normalized and satisfy the resolution of the identity in \mathcal{H} with

$$\nu(dx) = \mathcal{N}(x) \mu(dx). \quad (5)$$

3 Coherent states for a massive particle in 1+1 dS space

The above construction is now applied to the phase space of a massive free particle in a 1 + 1 de Sitter space. This can be realized as follows [4]. The phase space reads

$$X = T^*(S^1) = \left\{ (\vec{x}, \vec{J}) \in \mathbb{R}^2 \times \mathbb{R}^2 \mid x^2 = 1, \vec{x} \cdot \vec{J} = 0 \right\} \quad (6)$$

$$= \left\{ (\beta, J) \mid 0 \leq \beta < 2\pi, J \in \mathbb{R} \right\}, \quad (7)$$

where (β, J) are conjugate coordinates. Let $|n\rangle, n \in \mathbb{Z}$, be the state vector corresponding to the function

$$\Phi_n^\epsilon(\beta, J) = e^{-\frac{\epsilon n^2}{2}} e^{n(\epsilon J + i\beta)}, \quad n \in \mathbb{Z}. \quad (8)$$

The set $\{|n\rangle, n \in \mathbb{Z}\}$ is an orthonormal family of $L^2(X, \mu)$, where

$$\mu(d\beta, dJ) = \sqrt{\frac{\epsilon}{\pi}} \frac{1}{2\pi} e^{-\epsilon J^2} dJ d\beta. \quad (9)$$

This family fulfills the condition (3). As a consequence, coherent states can be defined on this phase space through [4, 5, 6, 7, 8, 9]:

$$|J, \beta\rangle = \frac{1}{\sqrt{\mathcal{N}_\epsilon(\beta, J)}} \sum_n (\Phi_n^\epsilon(\beta, J))^* |n\rangle, \quad (10)$$

where $\mathcal{N}_\epsilon(\beta, J)$ is the normalization factor given by the following convergent series ($\epsilon > 0$)

$$\begin{aligned}\mathcal{N}_\epsilon(\beta, J) = \mathcal{N}_\epsilon(J) : &= \sum_n |\Phi_n^\epsilon(\beta, J)|^2 \\ &= \sum_n e^{-\epsilon n^2 + 2\epsilon n J} < \infty.\end{aligned}\quad (11)$$

These coherent states lead to the following resolution of the identity in \mathcal{H}

$$\int_{J=-\infty}^{\infty} \int_{\beta=0}^{2\pi} |J, \beta\rangle \langle J, \beta| \mathcal{N}_\epsilon(\beta, J) \mu(d\beta, dJ) = \text{Id}_{\mathcal{H}}, \quad (12)$$

where \mathcal{H} is the Hilbert space spanned by the $|n\rangle$. We are now in position to quantize classical observables. For instance, applying (2) to $J^m, m \in \mathbb{N}$ and β leads to

$$O_{J^m} = \sum_n \left(\frac{i}{2\sqrt{\epsilon}} \right)^m H_m(-i\sqrt{\epsilon}n) |n\rangle \langle n|, \quad (13)$$

$$O_\beta = \pi \text{Id}_{\mathcal{H}} + i \sum_{n \neq n'} \frac{e^{-\frac{\epsilon(n-n')^2}{4}}}{n - n'} |n\rangle \langle n'|, \quad (14)$$

where H_m is the Hermite polynomial.

4 Ordering algebra

It is well known [10] that the contravariant quantization, in the context of the standard coherent states, corresponds to the anti-normal ordering. More precisely, in the Bargman representation $O_z = z$ and $O_{z^*} = \frac{\partial}{\partial z}$. Thus the upper symbol associated with the classical observable $f = \sum A_{nm}(z^*)^n z^m$ reads:

$$O_f = \sum A_{nm} \left(\frac{\partial}{\partial z} \right)^n z^m. \quad (15)$$

Something very similar appears in our context. Let f be a classical observable which admits the series representation $\sum_{p,q} c_{p,q} A^{*p} A^q$ $p, q \in \mathbb{Z}$, where $A = e^{+\epsilon J + i\beta}$. Then, f can be quantized as :

$$O_f = \sum_{p,q} c_{p,q} O_{A^{*p}} O_{A^q}, \quad p, q \in \mathbb{Z}, \quad (16)$$

where the A^* 's appear on the leftmost position. This is the main result of this letter. The proof is easy, here it is: matrix elements of operators associated to A^{*p} , A^q and $A^{*p}A^q$ are calculated using (2). It follows that

$$\begin{aligned} O_{A^{*p}} &= \sum_n e^{\frac{\varepsilon}{2}p(p+2n)} |n\rangle \langle n+p|, \\ O_{A^q} &= \sum_n e^{-\frac{\varepsilon}{2}q(q-2n)} |n\rangle \langle n-q|, \\ O_{A^{*p}A^q} &= \sum_n e^{\frac{\varepsilon}{2}(2pq-(p+q)(q-p-2n))} |n\rangle \langle n-(q-p)|. \end{aligned}$$

A straightforward calculation shows that

$$O_{A^{*p}} O_{A^q} = O_{A^{*p}A^q}, \quad (17)$$

from which (16) follows using linearity. Note that

$$\begin{aligned} O_{A^*} &= (O_A)^\dagger, \\ O_{A^{-1}} &= (O_A)^{-1}. \end{aligned}$$

The operator (16) is weakly defined as soon as p and q admit a lower bound. Indeed, the matrix elements of (16) read

$$\langle m | O_f | n \rangle = \sum_{\substack{p,q \\ p+q=m-n}} c_{pq} e^{\frac{\varepsilon}{2}(p(p-2m)+q(2n+q))},$$

which is a finite sum in this case.

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